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Automorphism groups of subalgebras; a concrete characterization

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FRIED and SICHLER [1] raise the following question: Let L be an algebraic lattice and let G_x be a group for every $x \in L$. Under what circumstances is there an algebra $\mathfrak{A}$ with $\mathrm{Sub}(\mathfrak{A}) \cong L$ such that if $\mathfrak{A}_x$ denotes the subalgebra corresponding to x, then $\mathrm{Aut}(\mathfrak{A}_x) \cong G_x$. In this note we solve the concrete version of the problem (Theorem 1).

The proof uses the techniques developed by B. JÓNSSON [3], M. G. STONE [4], J. JEŽEK [2], L. SZABÓ [5]. A direct application of our result to the original problem yields a new proof of Theorem 1 in [1] (Corollary 2).

In the second part of this note we obtain a partial result on the representability of a small concrete category as a category of universal algebras with prescribed subalgebras.

1. Let A be a set, L a family of subsets of A and let a permutation group G_R, acting on R, be assigned to each $R \in L$. The elements of G_R may then be regarded as partial mappings of A. Let φ and ψ be two partial mappings of A (into itself). Then we can define their product $\psi\varphi$, where $(\psi\varphi)a$ is defined iff $a \in \mathrm{Dom}\,\varphi$, $\varphi a \in \mathrm{Dom}\,\psi$ and in that case $(\psi\varphi)a = \psi(\varphi a)$. So, all the partial mappings of A, including the empty mapping $\emptyset \to \emptyset$ form a semigroup. Let $\bar{G}$ denote its subsemigroup, generated by $\{G_R : R \in L\}$.

We shall call G_R locally $\bar{G}$ closed if the following condition holds: For every permutation h of R, if for every finite subset X of R there exists a member of $\bar{G}$ that agrees with h on X, then $h \in G_R$.

For $\varphi : A \to A$ a partial injection (thus $x \neq y$ implies $\varphi x \neq \varphi y$ provided $x, y \in \mathrm{Dom}\,\varphi$), φ^{-1} can be defined in a natural way.

With the above notation, we have

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Theorem 1. *Let A be a set, L a family of subsets of A, and let a permutation group G_R, acting on R, be assigned to each $R \in L$. A universal algebra $\mathfrak{A} = \langle A, F \rangle$ with precisely the elements of L for its subuniverses, and with precisely the elements of G_R for the automorphism of $R \in L$, exists if and only if*

(i) *L is an algebraic closure system on A,*

(ii) *G_R is locally $\bar{G}$-closed for any $R \in L$,*

(iii) *$\sigma R \in L$ for any $\sigma \in \bar{G}$ and $R \in L$,*

(iv) *$\sigma | X = \tau | X$ implies $\sigma | \langle X \rangle = \tau | \langle X \rangle$ provided $\sigma, \tau \in \bar{G}$ and $X \subset A$ is finite.*

(Here $\langle X \rangle$ denotes the closure of X with respect to L.)

Proof. The conditions listed are necessary, since the elements of $\bar{G}$ are isomorphisms between certain subalgebras of A.

To prove sufficiency, let Y denote the set of the one-to-one sequences of finite length composed of the elements of A.

For any $y = (y_1, \ldots, y_n) \in Y$, $a \in \langle y_1, \ldots, y_n \rangle$ let us define an n-ary operation $f_{(y,a)}$ on A by

$$f_{(y,a)}(w_1, \ldots, w_n) = \begin{cases} \sigma a & \text{if } \exists \sigma \in \bar{G}, w = \sigma y \\ w_1 & \text{otherwise.} \end{cases}$$

(Note, in particular, that $f_{(y,a)}(y) = a (y \in A^n, a \in A)$.)

The definition makes sense owing to condition (iv) ($\sigma_1 y = \sigma_2 y$ implies $\sigma_1 a = \sigma_2 a$.). In this way we have constructed a universal algebra $\mathfrak{A} = \langle A, f_{(y,a)} : y \in Y, a \in \langle y \rangle \rangle$. We assert that $\mathfrak{A}$ complies with the requirements.

Let $R \in L$; $w_1, \ldots, w_n \in R$ and let us consider $f_{(y,a)}(w)$. If the second situation obtains, then $f_{(y,a)}(w) \in R$. Let therefore $w = \sigma y$. Then $\sigma^{-1} w = y$, whence $\langle w \rangle = \sigma \langle y \rangle$; but since $a \in \langle y \rangle$, we have $\sigma a \in \langle w \rangle \subset R$. Hence $R \in \mathrm{Sub}(\mathfrak{A})$.

Conversely let $Q \in \mathrm{Sub}(\mathfrak{A})$, and let $a \in \langle Q \rangle$. In that case there exist some $y_1, \ldots, y_n \in Q$ such that $a \in \langle y_1, \ldots, y_n \rangle$. But then $f_{(y,a)}(y) = a \in Q$. Consequently $Q = \langle Q \rangle$, implying that the subuniverses are precisely the elements of L.

Given $\tau \in \bar{G}$ and $w \in (\mathrm{dom}\, \tau)^n$, we assert that $f_{(y,a)}(\tau w) = \tau f_{(y,a)}(w)$. For, let $w = \sigma y$. Then $\tau w = \tau \sigma y$, whence $f_{(y,a)}(\tau w) = \tau \sigma a = \tau f_{(y,a)}(w)$. If $\tau w = \sigma y$, then $\tau w = \sigma y = \tau(\tau^{-1} \sigma) y$, whence $w = \tau^{-1} \sigma y$, implying $f_{(y,a)}(\tau w) = \tau (\tau^{-1} \sigma) a = \tau f_{(y,a)}(w)$. In particular, the elements of the G_R-s are automorphisms of $\mathfrak{A} | R$.

If $\varphi \notin G_R$, then by (ii) there exist $y_1, \ldots, y_n \in R$ such that $\varphi y \neq \sigma y$ for any $\sigma \in \bar{G}$, and we may assume that $n \geqq 2$. But then $\varphi f_{(y, y_2)}(y) = \varphi y_2$; $f_{(y, y_2)}(\varphi y) = \varphi y_1$, since, however, $y_1 \neq y_2$, also $\varphi y_2 \neq \varphi y_1$, whence $\varphi \notin \mathrm{Aut}(\mathfrak{A} | R)$ completing the proof.

2. Now we derive some corollaries of Theorem 1. Let $\mathfrak{A}$ be a universal algebra, $L = \mathrm{Sub}(\mathfrak{A})$, $G_x = \mathrm{Aut}(x) : x \in L$, $L_x = \{y \in L : y \subseteqq x\}$. Then there exist natural homomorphisms $\Phi_x : G_x \to \mathrm{Aut}(L_x)$.

Let $H_x = \mathrm{Ker}\, \Phi_x$. Let us first prove the following statement.

Corollary 1. *Given an algebra* $\langle A, F\rangle$ *there exists an algebra* $\langle A, G\rangle$ *such that* $\mathrm{Sub}\langle A, G\rangle=\mathrm{Sub}\langle A, F\rangle$ *and* $\mathrm{Aut}_G(x)=H_x$ *for any* $x\in\mathrm{Sub}\langle A, G\rangle$.

Proof. Of the conditions listed under Theorem 1, (i) clearly holds for the system $(L, H_x:x\in L)$; and so do (iii) and (iv), since $\bar{H}\subseteqq\bar{G}$. If the permutation $\varphi:x\to x$ coincides on any finite set with a permutation in $\bar{H}$, then it coincides with a permutation in $\bar{G}$, whence $\varphi\in G_x$. On the other hand, the elements of $\bar{H}$ leave any subuniverse in place. We conclude that φ belongs to H_x: hence, (ii) also holds, which completes the proof.

If now $\varphi\in H_x$, and $y\subseteqq x$. Then $\varphi|y\in H_y$, and we get a homomorphism $r_{xy}:H_x\to H_y$. Let $\Sigma=(L, (K_x:x\in L), (P_{xy}:x, y\in L, x\geqq y))$; where L is an algebraic lattice, K_x a group for each $x\in L$, and P_{xy} a homomorphism of K_x into K_y $(x, y\in L)$.

We say that Σ is representable if there exists an algebra $\mathfrak{A}$ such that $\mathrm{Sub}(\mathfrak{A})\cong L$, $K_x\cong H_x=G_x$ for each $x\in L$, each P_{xy} represents the restriction homomorphism r_{xy}.

Now we can prove the following

Corollary 2. (Fried—Sichler [1]) *Σ is representable iff*

(v) $P_{yx}\cdot P_{zy}=P_{zx}$ *for all* $x\leqq y\leqq z$,

(vi) $\mathrm{Ker}\,P_{xy}\cap\mathrm{Ker}\,P_{xz}$ *is trivial whenever* $x=y\cup z$,

(vii) *if* $x\in L$ *is not compact then* K_x *is the inverse limit of the diagram* $(P_{cd}:x>c\geqq d)$ *with the limit homomorphisms* P_{xc} $(c<x)$,

(viii) $K_0=1$.

Proof. The necessity of the conditions can be easily checked (cf. [1]). For the sufficiency we use the construction given in [1]. The correctness of the construction will easily follow from our Theorem 1.

Let $C=\{c\in L:c \text{ compact}\}$. $A=\{(c,\alpha):c\in C,\ \alpha\in K_c\}$. $A_x=\{(c,\alpha): c\leqq x,\ \alpha\in K_c\}$. $\bar{L}=\{A_x:x\in L\}$. Then $\bar{L}$ is an algebraic closure system, and regarded as a lattice it is isomorphic with L.

For any $\varphi\in K_x$ define $T_\varphi:A_x\to A_x$ by $T_\varphi(c,\alpha)=(c,(P_{xc}\varphi)\alpha)$. $G_x=\{T_\varphi:\varphi\in K_x\}$. Now $G_x\cong K_x$, and the elements of G_x leave the subuniverses in place, whence $G_x=H_x$.

Because of $G_x=H_x$ we have $\bar{G}=\bigcup G_x$, so (iii) holds. Now if X is a finite subset of A and $\sigma,\tau\in\bar{G}$ and $\sigma|X=\tau|X$ then for all $x\in X$ $\sigma|\langle x\rangle=\tau|\langle x\rangle$ by the definition of T_φ, hence by (vi) we also have $\sigma|\langle X\rangle=\tau|\langle X\rangle$. If now $A_x\in\bar{L}$, $\psi:A_x\to A_x$ and ψ coincides on any finite subset of A_x with an element of $\bar{G}$, then clearly we have $\psi(c,\alpha)=(c,\psi_c(\alpha))$ for some permutation ψ_c of K_c, since T_φ maps K_c into itself for all φ and c. Now considering our condition for the two element set $\{(c,1),(c,\alpha)\}$ we have $\psi(c,1)=T_\varphi(c,1)$, $\psi(c,\alpha)=T_\varphi(c,\alpha)$ for some T_φ and we have $\psi(c,\alpha)=(c,\psi_c(1)\alpha)$. If we consider $\{(c,1),(d,1)\}$ for any pair $c\leqq d$ then we have $\psi_c(1)=P_{dc}\psi_d(1)$, hence if $A_y \subset A_x$ and A_y is finitely generated then $\psi_y(1)$ determines uniquely $\psi|A_y$ hence $\psi|A_y=T_\varphi|A_y$ for some $T_\varphi \in G_y$. Applying (vii) on the system of the T_φ-s (as A_y runs over the set of finitely generated subalgebras of A_x) we get that $\psi=T_\sigma$ for some $\sigma \in G_x$. Hence Theorem 1 can be applied to prove corollary 2.

3. In [4] STONE considered the subalgebras and the automorphisms of subalgebras, in [5] SZABÓ (Theorem 1) the subalgebras and the isomorphisms between subalgebras, and in [2] JEŽEK (Theorem 2) considered a small category of algebras and the injective morphism. We are going to derive a common generalization of these results.

Let K be a small subcategory of SETS. The elements of Mor K can be considered as partial mapping of $\bigcup \mathrm{Ob}\, K$ (we consider this union to be disjoint). Let $\bar{S}$ denote the semigroup generated by the injective elements of Mor K and their inverses. With this notation we have

Theorem 2. *Let K be a small subcategory of* SETS *such that if $\alpha \in \mathrm{Mor}\, K$, then α is either injective or a mapping onto a single point. For each $A \in \mathrm{Ob}\, K$, let $L(A)$ be a family of subsets of A. There exists a set F of operation symbols and universal algebras $\langle A, F\rangle$ to each $A \in \mathrm{Ob}\, K$, such that* $\mathrm{Sub}\,\langle A, F\rangle = L(A)$ *and* $\mathrm{Hom}\,(\langle A, F\rangle, \langle B, F\rangle) = \mathrm{Mor}\,(A, B)$ *iff*

(ix) *$L(A)$ is an algebraic closure system for every $A \in \mathrm{Ob}\, K$,*

(x) *$R \in L(A)$, $\varphi: A \to B \in \bar{S}$, $R \subset \mathrm{Dom}\, \varphi$ implies $\varphi R \in L(B)$,*

(xi) *$\mathrm{Mor}\,(A, B)$ is locally $\bar{S}$ closed,*

(xii) *if $R \in L(A)$, $|R|=1$, $B \in \mathrm{Ob}\, K$ then $\varphi_{B,R}: B \to R \in \mathrm{Mor}\,(B, A)$,*

(xiii) *if $\sigma, \tau \in \bar{S}$, $X \subset A$ is finite then $\sigma|X=\tau|X$ implies $\sigma|\langle X\rangle=\tau|\langle X\rangle$ where $\langle X\rangle$ denotes the closure of X with respect to $L(A)$.*

Proof. Necessity of (ix), (x) and (xii) is obvious. The elements of $\bar{S}$ are compatible with the operations, therefore (xi) and (xiii) are also necessary.

In order to prove the sufficiency let $Y(A)$ denote the set of one-to-one sequences formed from the elements of A and $Y=\bigcup\{Y(A): A \in \mathrm{Ob}\, K\}$. For each $y=(y_1, \dots, y_n) \in Y$, $a \in \langle y_1, \dots, y_n\rangle$, we define an n-ary operation $f_{(y,a)}$: for $B \in \mathrm{Ob}\, K$, $w_1, \dots, w_n \in B$ set

$$f_{(y,a)}(w_1, \dots, w_n) = \begin{cases} \sigma a & \text{if there exists a } \sigma \in \bar{S} \text{ such that } w = \sigma y \\ w_1 & \text{otherwise.} \end{cases}$$

This definition makes sense owing to condition (xiii). Having endowed each $A \in \mathrm{Ob}\, K$ with this set $F=\{f_{(y,a)}\}$ of operations, we assert that the resulting set $\{\langle A, F\rangle: A \in \mathrm{Ob}\, K\}$ of universal algebras complies with our requirements.

As in Theorem 1, we can deduce that $\mathrm{Sub}\langle A, F\rangle = L(A)$, and $\mathrm{Mor}(A, B) \subseteq \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$. Therefore it suffices to prove that $\mathrm{Hom}(\langle A, B\rangle, \langle B, F\rangle) \subseteq \mathrm{Mor}(A, B)$.

Let $\varphi \in \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$. If $|\mathrm{Im}\,\varphi| = 1$ then $\varphi \in \mathrm{Mor}(A, B)$ because of (xii).

If φ is injective, and $y_1, \ldots, y_n \in A$ $(n \geqq 2)$, then $\varphi f_{(y, y_2)}(y) = \varphi y_2$, hence $f_{(y, y_2)}(\varphi y) = \varphi y_2$, but $\varphi y_1 \neq \varphi y_2$ and therefore $\varphi y = \sigma y$ for some $\sigma \in \bar{S}$ and we conclude from (xi) that $\varphi \in \mathrm{Mor}(A, B)$.

If φ is neither injective nor a mapping onto a single point, then there exist $y_1, y_2, y_3 \in A$ such that $\varphi y_1 \neq \varphi y_2 = \varphi y_3$. But then $\varphi y \neq \sigma y$ any $\sigma \in \bar{S}$, for the elements of $\bar{S}$ are injective and hence $\varphi f_{(y, y_2)}(y) = \varphi y_2 \neq \varphi y_1 = f_{(y, y_2)}(\varphi y)$. Hence $\varphi \notin \mathrm{Hom}(\langle A, F\rangle, \langle B, F\rangle)$, completing the proof.

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References

[1] E. Fried—J. Sichler, On automorphism groups of subalgebras of a universal algebra, *Acta Sci. Math.*, **40** (1978), 265—270.

[2] J. Ježek, Realization of small concrete categories by algebras and injective homomorphisms, *Coll. Math.*, **29** (1974), 61—69.

[3] B. Jónsson, Algebraic structures with prescribed automorphism group, *Coll. Math.*, **19** (1968), 1—4.

[4] M. G. Stone, Subalgebra and automorphism structure in universal algebras, a concrete characterization, *Acta Sci. Math.*, **33** (1972), 45—48.

[5] L. Szabó, Characterization of some related semigroups of universal algebras, *Acta Sci. Math.*, **37** (1975), 143—148.

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